Modeling Bond Prices In Continuous-Time
Part IV - Solving For Risky Bond Discount Rate

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November, 2020

In this white paper we will build a model that calculates the unknown market discount rate applicable to a risky bond with a known market value.

Our Hypothetical Problem

The table below presents our go-forward model assumptions from Part III...

Table 1: Risky Bond Assumptions

Symbol	Description	Balance
P_0	Market price at time zero	\$882.21
B	Bond face value	\$1,000.00
C	Annual coupon rate (%)	4.50
R	Recovery rate given a bond default (%)	40.00
D	Cumulative default rate (%)	5.00
S	Credit spread over the risk-free rate (%)	2.00
T	Term in years (#)	3.00

We are tasked with answering the following questions:

Question 1: What is the continuous-time discount rate applicable to this risky bond?

Question 2: What is the yield to maturity and bond equivalent yield?

Bond Price Equations From Part III

In Part III we defined the variable P_0 to be the price at time zero of a coupon paying risky bond and the variable κ to be the continuous-time discount rate. Using Table 1 above the equation for bond price at time zero is... [1]

$$P_0 = B\left[\left(C + \lambda\, R\right)\int\limits_0^T \mathrm{Exp}\left\{-(\kappa+\lambda)\,u\right\}\delta u + \mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\right] \tag{1}$$

The solution to Equation (1) above is... [1]

$$P_0 = B\left[\left(C + \lambda\, R\right)\left(\kappa+\lambda\right)^{-1}\left(1 - \mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\right) + \mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\right] \tag{2}$$

The equation for the first derivative of bond price with respect to discount rate from Part III is... [1]

$$\frac{\delta}{\delta\kappa}\,P_0 = B\left(\left(C+\lambda\,R\right)\frac{\delta}{\delta\kappa}\left(\kappa+\lambda\right)^{-1} - \left(C+\lambda\,R\right)\frac{\delta}{\delta\kappa}\mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\left(\kappa+\lambda\right)^{-1} + \frac{\delta}{\delta\kappa}\mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\right) \tag{3}$$

The solution to Equation (3) above from Part III is... [1]

$$\frac{\delta}{\delta\kappa}\,P_0 = -B\left[\left(C+\lambda\,R\right)\left(1 - \mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\left(1 + (\kappa+\lambda)\,T\right)\right)\left(\kappa+\lambda\right)^{-2} + T\,\mathrm{Exp}\left\{-(\kappa+\lambda)\,T\right\}\right] \tag{4}$$

Solving For The Discount Rate

We will define the variable r to be the actual discount rate (i.e. unknown to be solved for), the variable $\hat{r}$ to be a guess discount rate, the function $f(r)$ to be bond price at the actual discount rate (i.e. the observed bond price), the function $f(\hat{r})$ to be bond price at the guess discount rate, and the function $f'(\hat{r})$ to be the first derivative of bond price at the guess discount rate. Using these definitions we can solve for discount rate via the following Newton-Raphson method for solving nonlinear equations... [2]

$$\hat{r} + \frac{f(r) - f(\hat{r})}{f'(\hat{r})} = r + e \tag{5}$$

To solve for the actual discount rate we will come up with an initial guess rate and then iterate Equation (5) above until the error term e is zero (i.e. $r = \hat{r}$).

The Answer To Our Hypothetical Problem

Question 1: What is the continuous-time discount rate applicable to this risk-free bond?

Using Equations (2), (4) and (5) above the answer to our problem is...

Table 2: Newton-Raphson Solution

iteration	guess	f(guess)	f'(guess)	f(actual)		new guess
1	0.12000	790.30	-2166.560177	882.21	=	0.07758
2	0.07758	888.13	-2451.727344	882.21	=	0.07999
3	0.07999	882.23	-2434.511249	882.21	=	0.08000
4	0.08000	882.21	-2434.450522	882.21	=	0.08000
5	0.08000	882.21	-2434.450521	882.21	=	0.08000

The discount rate used by the market to price this bond is 8.00%. We started with a guess rate of 12.00% and the solution took less than five iterations of the Newton-Raphson method to arrive at the actual rate of 8.00%.

Question 2: What is the yield to maturity and bond equivalent yield?

Using the answer to the question above the yield to maturity for this bond is...

$$\text{YTM} = \text{Exp}\left\{\kappa\right\} - 1 = \text{Exp}\left\{0.08000\right\} - 1 = 8.33\% \tag{6}$$

Using Equation (6) above the bond equivalent yield for this bond is...

$$\text{BEY} = 2 \times \left((1 + \text{YTM})^{0.5} - 1\right) = 2 \times \left((1 + 0.0833)^{0.5} - 1\right) = 8.16\% \tag{7}$$

Note: The bond pays coupon payments semi annually.

References

[1] Gary Schurman, *Modeling Bond Price in Continuous-Time - Part III,* November, 2020.

[2] Gary Schurman, *Newton-Raphson Method for Solving Nonlinear Equations - Part I,* October, 2009.